

# UPPER BOUND ON THE MINIMAL NUMBER OF RAMIFIED PRIMES FOR ODD ORDER SOLVABLE GROUPS

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## Abstract

Let  $G$  be a finite group and let  $\text{ram}^t(G)$  denote the minimal positive integer  $n$  such that  $G$  can be realized as the Galois group of a tamely ramified extension of  $\mathbb{Q}$  ramified only at  $n$  finite primes. Let  $d(G)$  denote the minimal non negative integer for which there exists a subset  $X$  of  $G$  with  $d(G)$  elements such that the normal subgroup of  $G$  generated by  $X$  is all of  $G$ . It is known that  $d(G) \leq \text{ram}^t(G)$ . However, it is unknown whether or not every finite group  $G$  can be realized as a Galois group of a tamely ramified extension of  $\mathbb{Q}$  with exactly  $d(G)$  ramified primes. We will show that  $3 \cdot \log(|G|)$  is an upper bound for  $\text{ram}^t(G)$  for all odd order solvable group  $G$ .

## 1 Introduction

The inverse Galois problem states that for every group  $G$ , there should exist a Galois extension  $K/\mathbb{Q}$  with a Galois group which is isomorphic to  $G$ . It is evident that odd order groups are solvable groups, and we know [Ne.79] that every finite odd (solvable) group is realizable over  $\mathbb{Q}$ . A variant of the inverse Galois problem is the minimal ramification problem, which we will now discuss.

Let  $\text{ram}^t(G)$  denote the minimal positive integer  $n$  such that  $G$  can be realized as the Galois group of a tamely ramified extension of  $\mathbb{Q}$  ramified only at  $n$  finite primes, and let  $d(G)$  denote the minimal nonnegative integer for which there exists a subset  $X$  of  $G$  with  $d(G)$  elements such that the normal subgroup of  $G$  generated by  $X$  is all of  $G$ .

Let  $K/\mathbb{Q}$  be a finite Galois extension with Galois group  $G = \text{Gal}(K/\mathbb{Q})$ , where  $G$  is a finite group. Let  $p$  be a finite prime of  $\mathbb{Q}$ . If  $p$  ramifies in  $K$  and if  $\mathfrak{p}$  is a prime of  $K$  dividing  $p$ , then the inertia group  $I(\mathfrak{p} | p)$  is a nontrivial subgroup of  $G$ . If  $I$  is the subgroup of  $G$  generated by all  $I(\mathfrak{p} | p)$ , then the fixed field of  $I$  is an unramified extension of  $\mathbb{Q}$ . Since by Minkowski's theorem, there are no nontrivial unramified extensions of  $\mathbb{Q}$ , we must have that  $I = G$ . Suppose in addition that  $K/\mathbb{Q}$  is tamely ramified, i.e. for every prime  $p$  which ramifies in  $K$ , all of the inertia groups are cyclic and of order prime to  $p$ . In this case, let us denote  $I(\mathfrak{p} | p) = \langle g_p \rangle$  and we deduce that the normal subgroup of  $G$  generated by all of the  $g_p$  is  $G$ . We conclude that  $d(G) \leq \text{ram}^t(G)$ .

Whether or not every finite group can be realized as a Galois group of a tamely ramified extension of  $\mathbb{Q}$  with exactly  $d(G)$  ramified primes is an open question. The (tame) minimal ramification problem is the following: can

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every finite group  $G$  be realized as a Galois group of a tamely ramified extension of  $\mathbb{Q}$  with exactly  $d(G)$  ramified primes. In [Bo-Ma.10], Boston and Markin conjectured that the minimal ramification problem has a positive answer, namely that for every finite group  $G$  we have  $d(G) = \text{ram}^t(G)$ . Most of the known results are valid only for  $l$ -groups, where  $l$  is an odd prime. Moreover, most of these results give an upper bound on  $\text{ram}^t(G)$ . An important property of the bound is that one should be able to calculate it directly from the structure of  $G$  as an abstract group, namely the bound should not depend on the realization of the group as a Galois group. Serre [Se.92] noted, using the Scholz-Reichardt method, that for a finite  $l$ -group  $G$ ,  $|G| = l^n$ ,  $l$  being an odd prime:  $\text{ram}^t(G) \leq n$ . However,  $n = d(G)$  only if  $G$  is elementary abelian. Geyer and Jarden [Ge-Ja.98] generalized Serre's result for global fields. Namely, they showed that for a global field  $K$  with  $l \neq \text{char}(K)$  and  $\mu_l \not\subseteq K$  (note that  $l \neq 2$ ), there exists  $r = r(K)$  such that for every  $l$ -group  $G$  of order  $l^n$ :  $\text{ram}^t(G) \leq n + r$ . In particular, for  $K = \mathbb{Q}$  we have  $r = 0$ . Plans [Pl.04] sharpened Serre's upper bound over  $\mathbb{Q}$  by showing that the Scholz-Reichardt method yields the following bound:  $d(G) + \sum_{1 \leq i \leq n-2} d(G_i)$ , where  $G_i = C_i/C_{i+1}$ ,  $C_i$  being the descending central series of  $G$ , and for  $n \leq 2$  the sum equals zero. We deduce that the minimal ramification problem has a positive answer for all  $l$ -groups  $G$  of nilpotency class 2, where  $l$  is an odd prime. Kisilevsky, Neftin and Sonn [Ki-Ne-So.10] proved that the minimal ramification problem has a positive answer for the family of semiabelian<sup>1</sup> nilpotent groups. However, this family does not contain all finite nilpotent groups, for example it is shown in [De.95] that there are 10 groups of order 64 which are not semiabelian.

We will give an upper bound on the number of ramified primes for all finite odd order groups. In particular, we will prove the following theorem:

**Theorem 13.** *Let  $G$  be an odd order group, then:*

$$\text{ram}^t(G) \leq 3 \cdot \ln(|G|)$$

This upper bound is extracted from Neukirch's proof of the realization of odd order groups over number fields. Similar to previous results, the upper bound is in a form of a sum of ranks of a certain derived series (in this case it is the chief series) of the group.

## 2 Cohomology and Ramification in number fields.

Let  $K$  be a number field and denote  $G_K$  the absolute Galois group of  $K$  and let  $A$  be a  $G_K$ -module. As usual,  $H^q(K, A)$  is the cohomology group  $H^q(G_K, A)$ . Let  $K_{nr}$  be the maximal unramified extension of  $K$ .

We recall that the unramified cohomology group is defined to be the image of the following:

$$H^q(\text{Gal}(K_{p,nr}/K_p), A) \xrightarrow{\text{inf}} H^q(K_p, A)$$

where  $\mathfrak{p}$  is a prime of  $K$ . We denote  $H_{nr}^q(K_p, A) = \text{im}(\text{inf})$ . Consider the homomorphism:

$$H^q(K, A) \rightarrow \prod'_{\mathfrak{p}} H^q(K_p, A)$$

where the restricted product is taken with respect to  $H_{nr}^q(K_p, A)$ . We say that  $x \in H^q(K, A)$  is unramified at  $\mathfrak{p}$  if

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<sup>1</sup>See [Ne.11].

$x_{\mathfrak{p}} \in H_{nr}^q(K_{\mathfrak{p}}, A)$ , otherwise we say that  $x_{\mathfrak{p}}$  is ramified.

Let  $p$  be an odd prime number and assume that  $\text{char}(K) \neq p$ . Let  $\mu_p$  be the group of  $p$ -th roots of unity, assume that  $\mu_p \not\subseteq K$ , denote  $\mathfrak{K} = K(\mu_p)$  and let  $\Delta = \text{Gal}(\mathfrak{K}/K)$ . Let  $A$  be a finite  $\Delta$ -module and a trivial  $G_{\mathfrak{K}}$  module. An action of  $\delta \in \Delta$  on the group  $H^1(\mathfrak{K}, A)$  is then defined, namely:

$$(\delta \cdot \varphi)(\tau) = \delta(\varphi(\tau^{\delta})) = \delta(\varphi(\hat{\delta}^{-1}\tau\hat{\delta}))$$

where  $\tau \in G_{\mathfrak{K}}$ ,  $\varphi \in H^1(\mathfrak{K}, A)$ , and  $\hat{\delta} \in G_K$  is a lifting of  $\delta$ . A canonical character  $\theta : \Delta \rightarrow (\mathbb{Z}/p)^*$  is defined by  $\delta(\zeta) = \zeta^{\theta(\delta)}$ , where  $\delta \in \Delta$  and  $\zeta \in \mu_p$ . However, by abuse of notation we will write the action additively, namely,  $\delta(\zeta) = \theta(\delta)\zeta$ . Define the following elements in  $\mathbb{Z}/p[\Delta]$  by:

$$e_i = \frac{1}{|\Delta|} \sum_{\delta \in \Delta} \theta(\delta)^{-i} \delta$$

where  $i \in \mathbb{Z}$ . A simple calculation shows that  $e_i$  are idempotent elements. The action of  $e_1$  on  $\varphi \in H^1(\mathfrak{K}, \mu_p)$  is the following:

$$\begin{aligned} e_1 \cdot (\varphi)(\tau) &= \frac{1}{|\Delta|} \sum_{\delta \in \Delta} \theta(\delta)^{-1} (\delta \cdot \varphi)(\tau) = \frac{1}{|\Delta|} \sum_{\delta \in \Delta} \theta(\delta)^{-1} \delta(\varphi(\hat{\delta}^{-1}\tau\hat{\delta})) = \\ &= \frac{1}{|\Delta|} \sum_{\delta \in \Delta} \theta(\delta)^{-1} \theta(\delta) (\varphi(\hat{\delta}^{-1}\tau\hat{\delta})) = \frac{1}{|\Delta|} \sum_{\delta \in \Delta} \varphi(\hat{\delta}^{-1}\tau\hat{\delta}) \end{aligned}$$

A simple calculation shows that  $\gamma \in \Delta$  acts on  $e_1 \cdot \varphi \in e_1 \cdot H^1(\mathfrak{K}, \mu_p)$  as multiplication by  $\theta(\gamma)$ . A similar calculation yields that  $\gamma \in \Delta$  acts trivially on  $e_0 \cdot H^1(\mathfrak{K}, \mathbb{Z}/p)$ .

*Claim 1.* The following:

$$\begin{aligned} \psi^* : e_0 \cdot H^1(\mathfrak{K}, \mathbb{Z}/p) \otimes \mu_p &\rightarrow e_1 \cdot H^1(\mathfrak{K}, \mu_p) \\ e_0\varphi \otimes \zeta &\mapsto e_1(\psi\varphi) \end{aligned}$$

is a  $\Delta$ -module isomorphism, where  $\psi : \mathbb{Z}/p \rightarrow \mu_p$  is a fixed isomorphism.

*Proof.* Let us first notice the following:

$$e_1(\psi\varphi) = \frac{1}{|\Delta|} \sum_{\delta \in \Delta} \psi\varphi(\hat{\delta}^{-1}\tau\hat{\delta}) = \psi \left[ \frac{1}{|\Delta|} \sum_{\delta \in \Delta} \varphi(\hat{\delta}^{-1}\tau\hat{\delta}) \right] = \psi(e_0\varphi)$$

where  $\varphi \in H^1(\mathfrak{K}, \mathbb{Z}/p)$ . Now, assume that  $e_0\varphi_1 \otimes \zeta = e_0\varphi_2 \otimes \zeta$ . Thus  $e_0\varphi_1 = e_0\varphi_2$ , and we have:

$$\psi^*(e_0\varphi_1 \otimes \zeta) = e_1(\psi\varphi_1) = \psi(e_0\varphi_1) = \psi(e_0\varphi_2) = e_1(\psi\varphi_2) = \psi^*(e_0\varphi_2 \otimes \zeta)$$

Hence,  $\psi^*$  is well defined.  $\psi^*$  is clearly an homomorphism. Let  $e_0\varphi \otimes \zeta \in \ker(\psi^*)$ , then:

$$0 = \psi^*(e_0\varphi \otimes \zeta) = e_1(\psi\varphi) = \psi(e_0\varphi)$$

and the fact that  $\psi$  is an isomorphism, implies that  $e_0\varphi = 0$ , and thus  $\ker(\psi^*) = 0$ . The map  $\psi^*$  is also onto since for  $e\eta \in e_1 \cdot H^1(\mathfrak{K}, \mu_p)$ ,  $\psi^*(\psi^{-1}\eta) = e_1\eta$ . Last, we need to show that  $\psi^*$  respect the action of  $\Delta$ , indeed:

$$\begin{aligned} \delta(\psi^*(e_0\varphi \otimes \zeta)) &= \delta(e_1(\psi\varphi)) = \theta(\delta)e_1(\psi\varphi) = \theta(\delta)\psi^*(e_0\varphi \otimes \zeta) = \\ &= \psi^*(e_0\varphi \otimes \theta(\delta)\zeta) = \psi^*(\delta(e_0\varphi) \otimes \delta(\zeta)) = \psi^*(\delta(e_0\varphi \otimes \zeta)) \end{aligned}$$

□

**Corollary 2.**  $e_1 \cdot H^1(\mathfrak{K}, \mu_p)$  is isomorphic as a  $\Delta$ -module to  $H^1(\mathfrak{K}, \mathbb{Z}/p)^\Delta \otimes \mu_p$ .

*Proof.* We already showed that every  $\delta \in \Delta$  acts trivially on  $e_0 \cdot H^1(\mathfrak{K}, \mathbb{Z}/p)$ , thus It is clear that  $e_0 \cdot H^1(\mathfrak{K}, \mathbb{Z}/p)$  is contained  $H^1(\mathfrak{K}, \mathbb{Z}/p)^\Delta$ . In general, for a  $\mathbb{Z}/p[\Delta]$ -module  $A$ ,  $e_i A$  is the maximal submodule of  $A$  on which the elements  $\delta \in \Delta$  act as multiplication by  $\theta(\delta)^i$ . Hence we have equality. Combining this with the previous claim and we obtain the desired result. □

By Kummer Theory:

$$\begin{aligned} \mathfrak{K}^*/(\mathfrak{K}^*)^p &\longrightarrow H^1(\mathfrak{K}, \mu_p) \\ \alpha &\mapsto \varphi_\alpha \end{aligned}$$

where  $\varphi_\alpha(\sigma) = \sigma(\sqrt[p]{\alpha})/\sqrt[p]{\alpha}$ , is an isomorphism. This yields an isomorphism from  $e_1 \cdot \mathfrak{K}^*/(\mathfrak{K}^*)^p$  to  $e_1 \cdot H^1(\mathfrak{K}, \mu_p)$ .

From the five term exact sequence we get:

$$0 \longrightarrow H^1(\Delta, \mathbb{Z}/p) \longrightarrow H^1(K, \mathbb{Z}/p) \xrightarrow{res} H^1(\mathfrak{K}, \mathbb{Z}/p)^\Delta \longrightarrow H^2(\Delta, \mathbb{Z}/p) \longrightarrow H^2(K, \mathbb{Z}/p)$$

note that  $H^1(\Delta, \mathbb{Z}/p) = H^2(\Delta, \mathbb{Z}/p) = 0$  since the order of  $\Delta$  divides  $p-1$ , which is prime to  $p$ . Thus we obtain that the restriction map is an isomorphism.

Let us summarize our results in the following claim.

**Proposition 3.** *The following  $\Delta$ -modules:*

$$H^1(K, \mathbb{Z}/p) \otimes \mu_p \cong H^1(\mathfrak{K}, \mathbb{Z}/p)^\Delta \otimes \mu_p \cong e_1 \cdot H^1(\mathfrak{K}, \mu_p) \cong e_1 \cdot \mathfrak{K}^*/(\mathfrak{K}^*)^p$$

*are isomorphic.*

*Remark 4.* This claim is used without proof in [Ne.79, Main Lemma] where the field theoretical meaning is explained.

Let us denote the ideal in  $O_{\mathfrak{K}}$  associated to an element  $\alpha \in \mathfrak{K}^*/(\mathfrak{K}^*)^p$  as:

$$(\alpha) = \mathfrak{A}^p \cdot \mathfrak{P}_1^{a_1} \cdots \mathfrak{P}_j^{a_j}$$

where  $\mathfrak{A}$  is an ideal of  $O_{\mathfrak{K}}$ ,  $\mathfrak{P}_i$  is a prime ideal of  $O_{\mathfrak{K}}$ , and for all  $1 \leq i \leq j$  we have  $1 \leq a_i \leq p-1$ . A prime ideal  $\mathfrak{q}$  of  $K$  is called *relatively prime* to  $\alpha \in \mathfrak{K}^*/(\mathfrak{K}^*)^p$  if for every  $1 \leq i \leq j$  we have  $\mathfrak{P}_i \nmid \mathfrak{q}$ .

**Lemma 5.** *Let  $\mathfrak{q}$  be a prime of  $K$ . Assume that  $\mathfrak{q} \nmid p$  and that  $\mathfrak{q}$  is relatively prime to  $\alpha \in \mathfrak{K}^*/(\mathfrak{K}^*)^p$ . If  $x \otimes \zeta \in H^1(K, \mathbb{Z}/p) \otimes \mu_p$  is the element corresponding to  $\alpha$  from proposition 3 then  $x$  is not ramified at  $\mathfrak{q}$ .*

*Proof.* The element  $\alpha$  corresponds by Kummer theory to an extension  $\mathfrak{K}(\sqrt[p]{\alpha})/\mathfrak{K}$  using a character in  $H^1(\mathfrak{K}, \mu_p)$  (as shown explicitly by the isomorphism  $\mathfrak{K}^*/(\mathfrak{K}^*)^p \longrightarrow H^1(\mathfrak{K}, \mu_p)$ ). Proposition 3 shows us that this character actually comes from a character  $x \in H^1(K, \mathbb{Z}/p)$ . Let  $K_x$  denote the cyclic extension of  $K$  defined by  $x$ . This means that the compositum of  $K_x$  with  $\mathfrak{K}$  equals  $\mathfrak{K}(\sqrt[p]{\alpha})$ , namely:

$$\begin{array}{ccc} \mathfrak{K} \cdot K_x & = & \mathfrak{K}(\sqrt[p]{\alpha}) \\ | & & | \\ K_x & & \mathfrak{K} \\ \searrow & & \swarrow \\ & K & \end{array}$$

Let  $\mathfrak{q}$  be a prime of  $K$  that is relatively prime to  $\alpha$  and satisfies that  $\mathfrak{q} \nmid p$ . It is then evident that  $\mathfrak{q}$  is not ramified in  $\mathfrak{K}(\sqrt[p]{\alpha})$ . Since  $\mathfrak{K}(\sqrt[p]{\alpha}) = \mathfrak{K} \cdot K_x$  we deduce that  $\mathfrak{q}$  is also unramified in  $K_x$ . It is then clear that  $(K_x)_{\mathfrak{q}}$  is contained in  $K_{\mathfrak{q}, nr}$  and thus  $x_{\mathfrak{q}} \in H^1_{nr}(K_{\mathfrak{q}}, \mathbb{Z}/p)$ , namely,  $x$  is unramified at  $\mathfrak{q}$ .  $\square$

Let  $K \mid k$  be a Galois extension,  $\Omega \mid K$  an abelian extension, and assume that  $\mu_p \not\subseteq K$ . Let  $A$  be a trivial  $G_K$ -module with  $pA = 0$ ,  $S$  be a finite set of primes of  $K$ , and let  $y_{\mathfrak{p}} \in H^1(K_{\mathfrak{p}}, \mathbb{Z}/p)$  for  $\mathfrak{p} \in S$ . In [Ne.79, Main Lemma] we obtain an element  $x \in H^1(K, A)$  such that:

1.  $x_{\mathfrak{p}} = y_{\mathfrak{p}}$  for  $\mathfrak{p} \in S$ .
2. if  $\mathfrak{p} \notin S$  then  $x_{\mathfrak{p}}$  is cyclic, and if  $x_{\mathfrak{p}}$  is ramified then the prime  $\mathfrak{p}_0 = \mathfrak{p} \cap k$  of  $k$ , splits completely in  $\Omega$  and  $x_{\mathfrak{p}'} = 0$  for all primes  $\mathfrak{p}' \mid \mathfrak{p}_0$  of  $K$  different from  $\mathfrak{p}$ .

This element  $x \in H^1(K, A)$  is obtained as the corresponding element to a certain  $\alpha \in e_1 \cdot \mathfrak{K}^*/(\mathfrak{K}^*)^p$ . This element is calculated in [Ne.79, Appendix] and is the product of two elements  $\gamma \cdot \delta$ . We are interested in the properties of these elements outside of  $S$ . Indeed, it is shown that they satisfy the following conditions:

1.  $\delta_{\mathfrak{p}} \in U_{\mathfrak{p}} \mathfrak{K}_{\mathfrak{p}}^*/(\mathfrak{K}_{\mathfrak{p}}^*)^p$  for  $\mathfrak{p} \notin S \cup \{\mathfrak{q}\}$ , where  $\mathfrak{q}$  is a prime of  $\mathfrak{K}$  which is not in  $S$ .
2.  $\gamma_{\mathfrak{p}} \in U_{\mathfrak{p}} \mathfrak{K}_{\mathfrak{p}}^*/(\mathfrak{K}_{\mathfrak{p}}^*)^p$  for  $\mathfrak{p} \notin \{\mathfrak{p}_{i_1}, \mathfrak{p}_{i_2}\}$ , where  $\mathfrak{p}_{i_1}, \mathfrak{p}_{i_2}$  are primes of  $\mathfrak{K}$  which are not in  $S$ .

We are interested in understanding the element  $x \in H^1(K, A)$  which is obtained from the Main Lemma and corresponds to  $\alpha \in \mathfrak{K}^*/(\mathfrak{K}^*)^p$ . Let us first assume that  $A = \mathbb{Z}/p$ , and let  $S$  be a finite set of primes of  $K$ . Under the previous notation, for primes  $\mathfrak{p} \nmid p$  and  $\mathfrak{p} \notin S \cup \{\mathfrak{q}, \mathfrak{p}_{i_1}, \mathfrak{p}_{i_2}\}$  we know that  $\alpha_{\mathfrak{p}} = \delta_{\mathfrak{p}} \gamma_{\mathfrak{p}}$  is a unit in  $\mathfrak{K}_{\mathfrak{p}}^*/(\mathfrak{K}_{\mathfrak{p}}^*)^p$  and thus relatively prime to  $\alpha$ . Hence, from lemma 5 we deduce that  $x$  is not ramified at  $\mathfrak{p}$ . We deduce that  $x$  can be ramified outside of  $S$  at no more than 3 primes. The general case is done by induction over  $\dim_{\mathbb{Z}/p} A$ . For  $x \in H^1(K, A)$  we have the following decomposition:  $x = x' + x''$ , where  $x' \in H^1(K, A')$ ,  $x'' \in H^1(K, \mathbb{Z}/p)$ , and  $A = A' \oplus \mathbb{Z}/p$ . Similarly, we have  $x_{\mathfrak{p}} = x'_{\mathfrak{p}} + x''_{\mathfrak{p}}$  for  $x_{\mathfrak{p}} \in H^1(K_{\mathfrak{p}}, A)$ . In the proof of the [Ne.79, Main Lemma] it is shown how to find such  $x'_{\mathfrak{p}}, x''_{\mathfrak{p}}$  with properties which we will soon describe. Let  $V$  denote a finite set of primes of  $K$  which is closed under conjugation over  $k$  and contains the set  $S \cup \{\mathfrak{p} \mid x'_{\mathfrak{p}} \text{ is ramified}\}$ . Then we have:

1.  $x_{\mathfrak{p}} = x'_{\mathfrak{p}} + x''_{\mathfrak{p}} = x'_{\mathfrak{p}}$  for  $\mathfrak{p} \in V - S$ .
2.  $x_{\mathfrak{p}} = x'_{\mathfrak{p}} + x''_{\mathfrak{p}} = x''_{\mathfrak{p}}$  for  $\mathfrak{p} \notin V$ .

It is then clear that  $x$  is ramified outside of  $S$  at the same places  $x'$  and  $x''$  are ramified.  $x''$  is obtained in the same way as the special case where  $A = \mathbb{Z}/p$ , namely at most 3 primes outside of  $S$ .  $x'$  is obtained by induction and it is evident that it is ramified outside of  $S$  at no more than  $3 \cdot \text{rank}(A')$  primes. To conclude,  $x \in H^1(K, A)$  is ramified outside of  $S$  at no more than  $3 \cdot \text{rank}(A)$  primes. Let us conclude what we have just shown in the following proposition:

**Proposition 6.** *Under the previous notations,  $x \in H^1(K, A)$  is ramified outside of  $S$  at no more than  $3 \cdot \text{rank}(A)$  primes.*

We will need to work with elements of  $H^1(k, A)$  rather than elements of  $H^1(K, A)$  and for this we have the following theorem.

**Theorem 7.** [Ne.79] *Let  $A$  be a simple  $G_k$ -module with  $pA = 0$ . Let  $K | k$  be a Galois extension such that<sup>2</sup>  $k(A) \subseteq K$  but  $\mu_p \not\subseteq K$  and let  $\Omega | K$  be an abelian extension. Let  $S$  be a finite set of primes of  $k$  and  $y_{\mathfrak{p}} \in H^1(k_{\mathfrak{p}}, A)$  for  $\mathfrak{p} \in S$ . Then there exists an element  $x \in H^1(k, A)$  such that*

- 1)  $x_{\mathfrak{p}} = y_{\mathfrak{p}}$  for  $\mathfrak{p} \in S$ .
- 2) If  $\mathfrak{p} \notin S$ , then  $x_{\mathfrak{p}}$  is cyclic and if  $x_{\mathfrak{p}}$  is ramified then  $\mathfrak{p}$  splits completely in  $\Omega$ .

*Claim 8.* The element  $x \in H^1(k, A)$ , which is obtained in Theorem 7, is ramified at no more than  $3 \cdot \text{rank}(A)$  primes which are not in  $S$ .

*Proof.* The element is given as the sum of two other elements:  $x = \zeta + z \in H^1(k, A)$ . Let  $V = S \cup \{\mathfrak{p} \mid z_{\mathfrak{p}} \text{ is ramified}\}$ . In the proof of Theorem 7 it is shown that if  $\mathfrak{p} \in V - S$  then  $x_{\mathfrak{p}} = 0$ , namely  $x$  is unramified at  $\mathfrak{p}$ . If  $x_{\mathfrak{p}}$  is ramified for  $\mathfrak{p} \notin V$  then since  $z_{\mathfrak{p}}$  is unramified we must have that  $\zeta_{\mathfrak{p}}$  is ramified. The element  $\zeta$  is given as the image of an element  $\bar{\zeta} \in H^1(K, A)$  which is the element obtained in proposition 6. Namely,  $\zeta$  can ramify at no more than  $3 \cdot \text{rank}(A)$  primes outside of  $S$ , and thus  $x$  can be ramified at no more than  $3 \cdot \text{rank}(A)$  primes which are not in  $S$ .  $\square$

### 3 Upper bound for all odd order groups.

Let  $G$  and  $G'$  be profinite groups with a given homomorphisms  $f, f'$  into a finite group  $\Gamma$ . We consider all  $\psi \in \text{Hom}(G, G')$  for which the following diagram:

$$\begin{array}{ccc} G & \xrightarrow{\psi} & G' \\ \downarrow & \swarrow & \\ & \Gamma & \end{array}$$

commutes. Let  $\psi_1, \psi_2$  be two such elements, they are called equivalent if there exists an element  $a \in \ker(f')$  such that:

$$\forall x \in G \quad \psi_2(x) = a\psi_1(x)a^{-1}$$

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<sup>2</sup>For a finite  $G_k$ -module  $A$  we denote by  $k(A) | k$  the smallest extension of  $k$  for which  $A$  is a trivial  $G_{k(A)}$ -module.

namely if  $\psi_1$  and  $\psi_2$  are conjugates by some element of  $\ker(f')$ . The equivalence classes are denoted by  $[\psi]$  and  $\mathcal{HOM}_\Gamma(G, G')$  denotes the set of all equivalence classes. Moreover, the set of surjective representatives is denoted by  $\mathcal{HOM}_\Gamma(G, G')_{sur}$ . Let  $[\psi] \in \mathcal{HOM}_\Gamma(G_k, G)$  and let us consider the canonical restriction map:

$$\mathcal{HOM}_\Gamma(G_k, G) \rightarrow \prod_{\mathfrak{p}} \mathcal{HOM}_\Gamma(G_{k_p}, G)$$

An element  $[\psi] \in \mathcal{HOM}_\Gamma(G_k, G)$  is called unramified at  $\mathfrak{p}$  if the inertia group  $I_{k_p}$  of  $G_{k_p}$  is contained in the kernel of  $\psi_p$ , where  $[\psi_p] \in \mathcal{HOM}_\Gamma(G_{k_p}, G)$  is the corresponding element.

Let  $k$  be an algebraic number field. Assume that  $\varphi$  is an homomorphism of  $G_k$  into the finite group  $\Gamma$ , and denote by  $K$  the kernel of  $\varphi$ . Let us look at the following embedding problem:

$$\begin{array}{ccc} & \mathfrak{G} & \\ & \swarrow \downarrow \varphi & \\ 1 \rightarrow A \rightarrow G & \rightarrow & \Gamma \rightarrow 1 \end{array}$$

where  $A$  is an elementary abelian group. Let  $E \rightarrow G$  be a surjective homomorphism with a solvable kernel of exponent  $e$  and let  $n$  be a multiple of  $ep$ .

**Lemma 9.** [Ne.79] Let  $S$  be an arbitrary set of primes of  $k$  and assume<sup>3</sup> that  $(n, m(K)) = 1$ . If  $\prod_{\mathfrak{p}} \mathcal{HOM}_\Gamma(G_{k_p}, G) \neq \phi$ , then there exists an element  $\psi \in \mathcal{HOM}_\Gamma(G_k, G)_{sur}$  with the following properties:

- 1)  $\psi$  induces given elements  $\psi_p \in \mathcal{HOM}_\Gamma(G_{k_p}, G)$  at primes  $\mathfrak{p} \in S$ .
- 2) If  $\mathfrak{p}$  is a prime of  $k$  which is not in  $S$  and is unramified in  $K | k$ , then  $\mathcal{HOM}_G(G_{k_p}, E) \neq \phi$ .
- 3) For the field  $N$  defined by  $\psi : G_k \rightarrow G$  we have  $(n, m(N)) = 1$ .

*Remark 10.* The group  $E$  in the above lemma is needed in order to keep solving the embedding problems.

**Lemma 11.** Under the above notation, the element obtained in Lemma 9 is ramified outside of  $S$  at no more than  $3 \cdot \text{rank}(A)$  primes.

*Proof.* By assumption  $\prod_{\mathfrak{p}} \mathcal{HOM}_\Gamma(G_{k_p}, G) \neq \phi$  then by [Ne.79, Lemma 4], the set  $\mathcal{HOM}_\Gamma(G_k, G) \neq \phi$  so we can start with  $[\psi_0] \in \mathcal{HOM}_\Gamma(G_k, G)$ . Denote by  $N_0 | k$  the field defined by  $\psi_0$  and let  $\Omega = N_0(\zeta_n)$ , where  $\zeta_n$  denotes a primitive  $n$ -th root of unity. Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_r$  be primes outside of  $S$  for which  $\psi_0$  is ramified, and denote  $S^* = S \cup \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ . Thus, the primes  $\mathfrak{p} \in S^* - S$  are unramified in  $K | k$ , namely the homomorphism  $\varphi_p$  is unramified, and therefore can be lifted<sup>4</sup> to an unramified  $\Gamma$ -homomorphism  $\tilde{\psi}_p$ . For each  $\mathfrak{p} \in S^*$  let  $y_p \in H^1(G_{k_p}, \mathbb{Z}/p)$  be the cohomology class which sends  $[(\psi_0)_p]$  into  $[\tilde{\psi}_p]$ , namely:

$$[(\psi_0)_p]^{y_p} = [\tilde{\psi}_p]$$

where  $\tilde{\psi}_p$  are the elements which are given in advance. We now apply Theorem 7 and obtain an element  $x \in H^1(\mathfrak{G}, A)$  such that:

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<sup>3</sup> $m(K)$  denotes the number of roots of unity in  $K$ .

<sup>4</sup>[Ne.79, Lemma 5]

1)  $x_{\mathfrak{p}} = y_{\mathfrak{p}}$  for  $\mathfrak{p} \in S^*$ .

2) if  $\mathfrak{p} \notin S^*$  then  $x_{\mathfrak{p}}$  is cyclic and if  $x_{\mathfrak{p}}$  is ramified then  $\mathfrak{p}$  splits completely in  $\Omega$ .

The element  $x$  changes the solution of the embedding problem into a solution which is ramified outside  $S$  at no more than  $3 \cdot \text{rank}(A)$  primes. Namely,  $[\psi]$  which is given by:

$$[\psi] = [\psi_0]^x$$

has the desired property. Indeed, if  $\mathfrak{p} \in S^* - S$  then  $[\psi_{\mathfrak{p}}] = [\tilde{\psi}_{\mathfrak{p}}]$  which we know is unramified. Let  $\mathfrak{p} \notin S^*$  and assume that  $[\psi_{\mathfrak{p}}]$  is ramified. However,  $[\psi_{\mathfrak{p}}] = [\psi_0]^{x_{\mathfrak{p}}}$  and we know that  $[(\psi_o)_{\mathfrak{p}}]$  is unramified outside  $S^*$  so we must have that the cohomology class  $x_{\mathfrak{p}}$  is ramified. The lemma now follows from claim 8.  $\square$

Let us recall that a *chief series* is a maximal normal series of a group. Namely, a chief series of a group  $G$  is a finite collection of normal subgroups  $N_i \subseteq G$ :

$$G = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_t = \{1\}$$

such that each quotient group  $N_{i-1}/N_i$  is a minimal normal subgroup of  $G/N_i$ .

*Remark 12.* Note that the chief factors of every chief series are elementary abelian  $l$ -group, where  $l$  is a prime number. Indeed, for a solvable group  $G$ , let  $\{G_i\}_{i=1}^m$  denote the chief series. Then  $G_{m-1}$  is a minimal normal subgroup of  $G$  and thus is elementary abelian  $l$ -group. The remark now follows by induction and the fact that  $\{G_i/G_{m-1}\}_{i=1}^{m-1}$  form a chief series of the solvable group  $G/G_{m-1}$ . Lets denote by  $p_i$  the prime associated to the group  $N_{i-1}/N_i$ . Note that:

$$\sum_{i=1}^t \text{rank}(N_{i-1}/N_i) = \sum_{i=1}^t \log_{p_i}(|N_{i-1}/N_i|) \leq \sum_{i=1}^t \ln(|N_{i-1}/N_i|) = \ln\left(\prod_{i=1}^t |N_{i-1}/N_i|\right) = \ln(|G|)$$

**Theorem 13.** *Let  $G$  be an odd order group, then:*

$$\text{ram}^t(G) \leq 3 \cdot \ln(|G|)$$

*Proof.* Let us take a chief series of  $G$ :

$$G = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_t = \{1\}$$

and let  $G_i = G/N_i$ . The kernel of the natural map  $\pi_i : G_i \rightarrow G_{i-1}$  will be denoted by  $A_i = N_{i-1}/N_i$ . From remark 12 we know that each chief factor  $A_i$  is elementary abelian.

The following diagram describes our embedding problems:



$$\begin{array}{ccc}
& & G_k \\
& \psi_i \quad \cdot \cdot \cdot & \downarrow \quad \psi_{i-1} \\
& \cdot \cdot \cdot & \\
1 \rightarrow A_i \rightarrow G_i & \xrightarrow{\pi_i} & G_{i-1} \rightarrow 1
\end{array}$$

We will review the construction of an element  $[\psi_i] \in \mathcal{HOM}_{\{e\}}(G_k, G)$  and count the number of ramified primes in the corresponding field. The construction is by induction on the chief series length, namely, it is shown that for each  $i = 1, \dots, t$  there exists an epimorphism  $\psi_i : G_k \rightarrow G_i$  with the following properties:

- (a)  $\psi_{i-1} = \pi_i \circ \psi_i$ ,  $i = 1, 2, \dots, t$ .
- (b)  $[\psi_i | G_{k_p}] = [\psi_p]$  in  $\mathcal{HOM}_{\{e\}}(G_{k_p}, G_i)$  for  $p \in S$ .
- (c) For the field  $K_i$ , defined by  $\psi_i$ , we have  $(|G|, m(K_i)) = 1$ .
- (d)  $\prod_p \mathcal{HOM}_{G_i}(G_{k_p}, G) \neq 0$ .

These properties are essential in order to keep solving the embedding problems. For  $i = 0$  we simply take  $\psi_0$  to be the trivial map. Let us assume that the homomorphism  $\psi_{i-1}$  is already constructed. Let  $S_{i-1}$  be the set of all primes of  $\mathbb{Q}$  which ramify in the field  $K_{i-1}$  which corresponds to  $\psi_{i-1}$ . In [Ne.79] it is shown that we can then apply Lemma 9 to this situation by replacing:  $S$  by  $S_{i-1}$ ,  $E \rightarrow G$  by  $G \rightarrow G_i$  and  $\varphi$  by  $\psi_{i-1}$ . We then obtain an element  $\psi_i$ , which is a representative of  $[\psi_i] \in \mathcal{HOM}_{G_{i-1}}(G_k, G_i)_{sur}$ , with the following properties:

- (i)  $[\psi_i | G_{k_p}] = [(\psi_i)_p]$  in  $\mathcal{HOM}_{G_{i-1}}(G_{k_p}, G_i)$  for  $p \in S_{i-1}$ .
- (ii) for  $p \notin S_{i-1}$  we have  $\mathcal{HOM}_{G_i}(G_{k_p}, G) \neq 0$ .
- (iii) For the field  $K_i$ , defined by  $\psi_i$ , we have  $(|G|, m(K_i)) = 1$ .

It is shown that  $\psi_i$  satisfy (a)-(d). From Lemma 11 we deduce that the element  $\psi_i$  is ramified outside of  $S_{i-1}$  at no more than  $3 \cdot \text{rank}(A_i)$  primes. Since  $S_{i-1}$  is defined to be the set of ramified primes in  $K_{i-1}$  we deduce that  $K_i$  has at most  $3 \cdot \text{rank}(A_i)$  more ramified primes over  $\mathbb{Q}$  than  $K_{i-1}$ .

We deduce that the number of ramified primes in the desired field is at most the sum of the “new” ramified primes in each step of the induction,  $\sum_{i=1}^t 3 \cdot \text{rank}(A_i)$ . According to Remark 12 we have:

$$\sum_{i=1}^t \text{rank}(A_i) \leq \ln(|G|)$$

and thus:

$$\text{ram}^t(G) \leq \sum_{i=1}^t 3 \cdot \text{rank}(A_i) \leq 3 \cdot \ln(|G|)$$

□

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